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A DYNAMIC LOCATION PROBLEM FOR GRAPHS

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We introduce a class of optimization problems, called *dynamic location problems*, involving the processing of requests that occur sequentially at the nodes of a graph G. This leads to the definition of a new parameter of graphs, called the window index $WX(G)$, that measures how large a "window" into the future is needed to solve every instance of the dynamic location problem on G optimally on-line. We completely characterize this parameter: $WX(G) \leq k$ if and only if G is a weak retract of a product of complete graphs of size at most k. As a byproduct, we obtain two (polynomially recognizable) structural characterizations of such graphs, extending a result of Bandelt.

1. Introduction

A wide range of optimization problems involve sequential decision-making in response to an external (i.e., uncontrolled) sequence of events. In such a system, the decisions do not affect the events themselves but the cost incurred for a chosen decision sequence depends in some way on the external events. A decision-making procedure can be viewed as an algorithm that takes as "input" the sequence of external events and produces as "output" the sequence of decisions. Quite often, these decisions must be produced "on-line", that is, each decision must be made based on currently available information: the history of events that have already occurred and incomplete knowledge of future events. A typical approach to such problems is to make probabilistic assumptions about the future input and to evaluate decision strategies based on their expected cost. In the absence of a good stochastic model for the input, an alternative approach for evaluating an on-line algorithm is to look at the maximum deviation of its cost on any input sequence from the optimal cost attainable (off-line) for that input sequence. This approach has been taken by a number of researchers ([9], [8], [21], [44], [45]), particularly in the context of data storage and retrieval. Of course the most desirable situation is for the on-line algorithm to be optimal on every input sequence, but one expects this not to be the case for most problems.

In this paper we introduce a simple and natural class of optimization problems, *dynamic location problems,* involving the processing of requests that occur sequentially at the nodes of a graph. The above considerations lead us to define a new parameter of a graph G, denoted by $WX(G)$, and called the *window index or*

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windex of G, that measures how large a "window" into the future is needed to solve every instance of the dynamic location problem on G optimally on-line. We obtain a complete characterization of $WX(G)$; our main result is that $WX(G)$ is finite if and only if G is a weak retract of a product of complete graphs. As a byproduct, we obtain two (polynomially recognizable) structural characterizations of such graphs, extending a theorem of Bandelt [4] on graph retracts (see also [16]).

2. The windex of a graph G

We consider the following simple model for the sequential processing of requests to a system. The system is represented by a connected graph $G=(V, E)$ where the vertices V represent "locations". The processing of each request requires the use of the same indivisible resource. The state of the system is characterized by a single vertex representing the current location of the resource. The state can be changed (that is, the resource moved) from v to w at a cost $d_G(v, w)$ where d_G denotes the standard graph distance (i.e., the shortest path metric). Requests arrive to the system at particular nodes and must be processed in the order of arrival. The cost of processing a request at node v is $d_G(v, w)$ where w is the location of the resource when the request is processed. A sequence of requests to the system is represented simply as a sequence $r_1 r_2 \dots r_n$ of nodes. The cost of processing the sequence $r_1 r_2 \dots r_n$ depends on the sequence $s_1 s_2 \dots s_n$ of states in which the resource resides when they are processed, and is equal to the sum of the cost of processing each request and the cost moving the resource between states, i.e.,

$$\sum_{i=1}^{n} (d_G(s_i, r_i) + d_G(s_{i-1}, s_i))$$

where s_0 is the (given) initial state of the system.

The *dynamic location problem* (DLP) for the graph G is to find the optimal sequence of states (minimizing the total cost) for a given request sequence. Given a particular finite request sequence, an optimal state sequence can be computed by standard dynamic programming techniques ([42]). This is an "off-line" computation, i.e., the choice of each state may depend on the entire request sequence. Suppose, however, that we are presented with the requests one at a time in order and we would like to be able to construct our state sequence on-line, i.e., to select each state s_i "soon after" finding out request r_i. More precisely we say that an algorithm for the DLP on G *works within window* k if for each i, the choice for state s_i depends only on s_0 and r_j for $j<i+k$. For each graph we ask: what is the smallest k (if any) such that the DLP has an optimal algorithm that works within window k? We call this parameter the *window index*, or *windex* of G, and denote it by $WX(G)$; if no such k exists we say $WX(G)=\infty$.

It is not immediately apparent that $WX(G)$ is a nontrivial parameter, i.e., that $WX(G)$ is not constant, or even that it is ever finite. Consideration of some examples, however, suggested that the parameter was interesting. Our early results (which will also follow from our main results) were:

Proposition 2.1. *$WX(G) \geqq 2$ for any connected graph on at least two vertices.*

Proposition 2.2. *The following classes of graphs have windex 2:*

a) *trees;*
b) *hypercubes;*
c) *planar grids.*

Proposition 2.3. *The complete graph K_n has windex n.*

Proposition 2.4 *The following graphs have windex ∞:*

a) *a cycle of 5 vertices;*
b) *K_4 with an edge deleted.*

For the classes of graphs listed in Proposition 2.2 the following window 2 algorithm turns out to be optimal: Having chosen state s_{i-1}, let s_i be a vertex that minimizes the sum of the distances from s_i to the three vertices s_{i-1}, r_i and r_{i+1}. The fact that this algorithm is optimal for these graphs stems from the following property that they share.

Unique Steiner Triple Property (USTP): *For any three (not necessarily distinct) vertices a, b, c there is a unique vertex v that minimizes the sum $d_G(a, v)+d_G(b, v)+$ $+d_G(c, v)$.*

Hence in the preceding window 2 algorithm the vertex s_i is uniquely determined from s_{i-1}, r_i and r_{i+1}.

In fact, as we showed in [12], this property characterizes graphs with windex 2:

Theorem 2.5. *$WX(G)=2$ if and only if G satisfies USTP.*

Characterizing graphs of windex greater than 2 requires another approach. The first step is the observation that windex behaves nicely under the operation of $\Box$-product, defined as follows.

If $G_i=(W_i, E_i)$, $1 \leqq i \leqq r$, are graphs then the $\Box$*-product* $G=G_1\Box G_2\Box \dots \Box G_r$ is the graph on vertex set $W_1\times W_2\times \dots \times W_r$ with edges $\langle (v_1, v_2, \dots, v_r), (w_1, w_2, \dots, w_r)\rangle$ if for some index j, $\langle v_j, w_j\rangle \in E_j$ and for $i\neq j$, $v_i=w_i$.

Lemma 2.6. *If $G_1, G_2, \dots, G_r$ are graphs then*

$$WX(G_1\Box G_2\Box \dots \Box G_r) = \max \{WX(G_1), WX(G_2), \dots, WX(G_r)\}.$$

The second step was to understand the relationship of $WX(G)$ to $WX(H)$ if H is a subgraph of G. It is natural to hope that if H is an induced subgraph of G then $WX(H)\leqq WX(G)$, but there are simple counterexamples; indeed $WX(G)$ may be finite while $WX(H)$ is infinite. Nevertheless, a weaker result is true:

Lemma 2.7. *If H is a weak retract of G (see section 3 for the definition) then $WX(H)\leqq$ $\leqq WX(G)$.*

Proposition 2.3, and Lemmas 2.6 and 2.7 provide a means for constructing a large class of graphs with finite windex: if G is a retract of a product of cliques then G has finite windex.

Now the following result of Hans Bandelt, brought to our attention by Peter Winkler, implies that this class contains all windex 2 graphs:

Theorem 2.8 [B] *G has the USTP if and only if G is a retract of a hypercube (i.e., a $\square$-product of complete graphs on two vertices).*

In fact, as we will show, this class contains all graphs of finite windex:

Theorem 2.9. *The following two conditions on a graph are equivalent:*

(i) *G is a weak retract of a product of cliques.*
(ii) *$WX(G)$ is finite.*
Furthermore, if $WX(G)$ is finite then $WX(G)=\Theta(G)$, the cardinality of the largest clique in G.

Along the way, we will prove two other equivalent characterizations of these graphs, given in Theorem 4.1, including a generalization of Theorem 2.8.

3. Notation and Other Preliminaries

A) Graphs

For a graph G, $V(G)$ and $E(G)$ denote the vertex and edge sets, respectively. Vertices of G are usually denoted by letters a, b, c, d, e, r, s. We usually use u, v, w, x, y, z to denote variables that range over vertices of G. An edge between a and b is denoted $\langle a, b\rangle$. For each $a\in V(G)$, the neighborhood of a, $N_G(a)$, is the set $\{v\in V|\langle a, v\rangle\in E(G)\}$. As with all functions defined with respect to G we will write simply $N(a)$ if there is no ambiguity. The distance between vertices a and b, $d_G(a, b)$ is the number of edges in the shortest path joining them. If $A, B\subseteqq V(G)$ then $d(A, B)=\min d(u, v)$ over all $u\in A$ and $v\in B$. $SP_G(a, b)$ is the set of all vertices (including a and b) that lie on some shortest path between a and b. Clearly $c\in SP(a, b)$ if and only if $d(a, c)+d(c, b)=d(a, b)$.

A median of a set of three vertices a, b, c in G has been defined elsewhere as a vertex x that lies on some shortest path between any two of them, i.e., such that

$$(3.1)\qquad d(a, b)+d(a, c)+d(b, c) = 2\big(d(a, x)+d(b, x)+d(c, x)\big).$$

Note that under this definition a, b, c cannot have a median if the sum of their distances is odd. For our purposes it will be convenient to use a modified definition and call x a *median* if

$$(3.2)\qquad d(a, b)+d(a, c)+d(b, c)+1 \geqq 2\big(d(a, x)+d(b, x)+d(c, x)\big).$$

and to say x is an *exact median* if (3.1) holds. (Note that if G is bipartite any median is exact.) If the median $x\notin \{a, b, c\}$, it is an *external median.*

A triple of vertices a, b and c is a *spike* if $d(a, c)=d(b, c)\geqq 2$ and either
(i) $\langle a, b\rangle\in E$
or
(ii) there is a vertex $e\in N(a)\cap N(b)$ such that $d(e, c)=d(a, c)+1$.

We say the spike is *type* 1 or *type* 2 depending on which of these conditions it satisfies. We have

Proposition 3.1. *If a, b, c is a spike with* $d(a, c)=d(b, c)=m$ *and x is an external median then* $x \in N(a) \cap N(b)$ *and* $d(x, c)=m-1$.

If $W \subseteq V(G)$ then $G_W=(W, E_W)$ denotes the *subgraph induced on G* containing all edges of G having both endpoints in W. An induced subgraph G_W is a *(weak) retract* of G (or W is a retract of V) if there exists a function $f: V \to W$ such that

(i) $f(v)=v$ for each $v \in W$

(ii) if $\langle v, w\rangle \in E(G)$ then either $f(v)=f(w)$ or $\langle f(v), f(w)\rangle \in E(G)$.

The map f is called a *(weak) retraction map*. A retraction map f is *strong* (and G_W is a *strong retract*) if in condition (ii) we do not allow $f(v)=f(w)$.

Two important properties of retracts are the following.

Proposition 3.2. *If f and g are retraction maps (resp. strong retraction maps) then so is their composition* $f \circ g$.

Proposition 3.3. *If f is a retraction map and* $v, w \in V$ *then* $d(v, w) \geqq d(f(v), f(w))$

(For background on retracts see [27], [28].)

An *isometric embedding* of a graph H into a graph G is a one-to-one function $\Theta: V(H) \to V(G)$ such that for any $v, w \in V(H)$, $d_H(v, w)=d_G(\Theta(v), \Theta(w))$. In particular Θ maps edges of H to edges of G and non-edges of H to non-edges of G. If H embeds isometrically into G we write $H \xrightarrow{I} G$.

Proposition 3.4. *If* $G_i=(W_i, E_i)$, $1 \leqq i \leqq r$, *are graphs, then for any*

$$(v_1, \ldots, v_r), (w_1, \ldots, w_r) \quad in \quad G = G_1 \square G_2 \square \ldots \square G_r,$$

$$d_G((v_1, \ldots, v_r), (w_1, \ldots, w_r)) = \sum_{i=1}^{r} d_G(v_i, w_i).$$

Elements of a product $W=W_1 \times \ldots \times W_r$ are denoted with a line over them, $\bar{v}=(v_1, v_2, \ldots, v_r)$, and are often called *vectors*. For any set V of vectors, B_V denotes the graph on vertex set V with $\langle \bar{v}, \bar{w}\rangle \in E(B_V)$ if $\bar{v}$ and $\bar{w}$ disagree in exactly one position. Such graphs are called *vector graphs*. If V is constant on some coordinate then deleting that coordinate does not change B_V; hence, in this case, every vector graph can be represented by a set of vectors having no constant coordinate, which we call an *irreducible* set.

Note that if V is equal to a product $W_1 \times W_2 \times \ldots \times W_r$ of sets then B_V is isomorphic to the $\square$-product of complete graphs on $W_1, W_2, \ldots, W_r$. In general, the *product completion* of a set of vectors V is the smallest set product $W=W_1 \times W_2 \times \ldots \times W_r$ containing V. Obviously, for each j, $W_j=\{a \mid \text{there is a } \bar{v} \in V \text{ with } v_j=a\}$, and B_V is an induced subgraph of B_W. Note that $d(\bar{v}, \bar{w})$ is always at least the number of coordinates in which $\bar{v}$ and $\bar{w}$ differ, with equality holding for all $\bar{v}$, $\bar{w}$ if and only if B_V is an isometric subgraph of B_W.

If $I=\{i_1, i_2, \ldots, i_j\} \subseteq \{1, \ldots, r\}$ the *projection* of $\bar{v}=(v_1, \ldots, v_n)$ onto I, denoted $proj_I(\bar{v})$, is the vector $(v_{i_1}, v_{i_2}, \ldots, v_{i_j})$ in $W_{i_1} \times W_{i_2} \times \ldots \times W_{i_j}$.

For $\bar{v}, \bar{w}, \bar{x} \in W$ we define the *imprint* of $\bar{v}$ and $\bar{w}$ on $\bar{x}$, $imp(\bar{v}, \bar{w}:\bar{x})$ to be the vector $\bar{z}$ defined by $z_i=v_i$ if $v_i=w_i$ and $z_i=x_i$ otherwise. A subset $V \subseteq W$ is *imprint closed* if $\bar{v}, \bar{w}, \bar{x} \in V$ implies $imp(\bar{v}, \bar{w}:\bar{x}) \in V$. If V is an imprint closed subset of W we also say that B_V is an imprint closed subgraph of B_W.

B) The dynamic location problem

A particular instance of the dynamic location problem for a graph $G=(V, E)$ is represented by a sequence $a_0 a_1 \dots a_k$ of vertices where a_0 is the initial state and $a_1 \dots a_k$ are the requests. We call such a sequence a DLP sequence. The *cost* of a response sequence $b_1 \dots b_k$ to $a_0 a_1 \dots a_k$ is defined to be

$$C(b_1 \dots b_k;\ a_0 a_1 \dots a_k) = d(b_1, a_0) + d(b_1, a_1) + \sum_{i=2}^{k} \big(d(b_{i-1}, b_i) + d(b_i, a_i)\big).$$

The cost of the optimal response sequence is

$$OPT(a_0 a_1 \dots a_k) = \min_{v_1 \dots v_k} C(v_1 \dots v_k;\ a_0 a_1 \dots a_k). \tag{3.3}$$

We will often denote a sequence of vertices by ϱ or σ. The concatenation of two sequences ϱ and σ is denoted $\varrho\sigma$.

It is easy to see that we can always take $v_k = a_k$ in the minimum above. Additional easy observations are:

Proposition 3.5.

(i) $OPT(a_0)=0$;
(ii) $OPT(a_0 a_1)=d(a_0, a_1)$;
(iii) $OPT(a_0 a_1 a_2)= \min_v [d(v, a_0)+d(v, a_1)+d(v, a_2)]$;
(iv) *For any sequences* $\varrho_1, \varrho_2, \dots, \varrho_k$, $OPT(\varrho_1 \varrho_2 \dots \varrho_k) \geqq OPT(\varrho_1)+OPT(\varrho_2)+ + \dots + OPT(\varrho_k)$.

From (iii) and (3.2) we have:

Proposition 3.6. $OPT(a_0 a_1 a_2) \geqq \left\lceil \frac{1}{2} (d(a_0, a_1)+d(a_0, a_2)+d(a_1, a_2)) \right\rceil$ *with equality if and only if* a_0, a_1, a_2 *have a median.*

By optimizing (3.3) first over some v_j we get:

Proposition 3.7. *If* $a_0, a_1, \dots, a_k \in V$ *and* $1 \leqq j \leqq k$ *then*

$$OPT(a_0 a_1 \dots a_k) = \min_{v \in V} d(v, a_j) + OPT(a_0 a_1 \dots a_{j-1} v) + OPT(v a_{j+1} \dots a_k).$$

In particular,

$$OPT(a_0 a_1 \dots a_k) = \min_{v \in V} \{d(a_0, v) + d(a_1, v) + OPT(v a_2 a_3 \dots a_k)\}.$$

If we use a window k algorithm to construct a state sequence $s_1 s_2 \dots s_k$ in response to a DLP sequence $a_0 a_1 a_2 \dots a_{k+1}$ then the first state s_1 depends only on $a_0 a_1 \dots a_k$. Hence, for the algorithm to be optimal that choice of s_1 must begin an optimal sequence for any value of a_{k+1}. Define, for $\varrho = a_0 a_1 \dots a_k$, the function

$$f_\varrho(u, v) = d(u, a_0) + d(u, a_1) + OPT(u a_2 \dots a_k v),$$

which is the minimum cost of a state sequence for $a_0 a_1 \dots a_k v$ given that $s_1 = u$. Note that for any v,

$$OPT(\varrho v) = \min_u f_\varrho(u, v).$$

The foregoing discussion implies:

Lemma 3.8. *If* $WX(G) \leqq k$ *then for any sequence* $\varrho = a_0 a_1 \dots a_k$ *there exists a vertex* b *such that for each* $v \in V$

$$f_\varrho(b, v) = \min_u f_\varrho(u, v) = OPT(\varrho v).$$

This simple lemma is the basis for all of our lower bounds on $WX(G)$,

4. The Main Theorem

We will prove the following refinement of Theorem 2.9.

Theorem 4.1. *The following conditions on a connected graph* $G = (V, E)$ *are equivalent:*

($P1$) *G has finite windex;*

($P2$) *G satisfies:*

(a) *If* $\langle u, v \rangle \notin E$ *then* $N(u) \cap N(v)$ *is either empty, a single vertex or two independent vertices;*

(b) *Every spike has an external median;*

($P3$) *G is isomorphic to an imprint closed subgraph of a product of cliques;*

($P4$) *G is isomorphic to a weak retract of a product of cliques.*

Furthermore, when these conditions hold, $WX(G) = \Theta(G)$, the maximum clique size of G.

The proof of Theorem 4.1 is organized as follows: $(P4) \to (P1)$ in Section 5, $(P1) \to (P2)$ in Section 6, $(P2) \to (P3)$ in Section 7, $(P3) \to (P4)$ in Section 8, and $WX(G) = \Theta(G)$ or ∞ in Section 9.

5. The Proof that (P4) → (P1)

It is enough to prove Proposition 2.3 and Lemmas 2.6 and 2.7.

Proof of Proposition 2.3. *We first show* $WX(K_n) > n-1$. *Let* $V = \{a_0, a_1, \dots, a_{n-1}\}$ *and let* $\varrho = a_0 a_1 \dots a_{n-1}$.

Claim: $f_p(u, a_0) \geqq n-1$ *with equality only if* $u = a_0$,
$f_p(u, a_1) \geqq n-1$ *with equality only if* $u = a_1$.

Then Lemma 3.8 implies that $WX(K_n) > n-1$.
To prove the claim we first note:

Lemma 5.1. *If* $b_1, \dots, b_k$ *are distinct then* $OPT(xb_1 \dots b_k) \geqq k-1$, *with equality if and only if* $x = b_i$ *for some* $1 \leqq i \leqq k$.

Proof. (By induction on k.) If $k=1$ then the result is trivial. If $k>1$, then by Proposition 3.7 and the induction hypothesis,

$$(5.1) \qquad OPT(xb_1 \dots b_k) = \min_y d(x, y) + d(b_1, y) + OPT(yb_2 \dots b_k) \geqq$$

$$\geqq \min_y d(x, y) + d(b_1, y) + k - 2$$

with equality if and only if $y \in \{b_2, \ldots, b_k\}$. Now for $y \in \{b_2, \ldots, b_k\}$ the right hand side of (5.1) is at least

$$d(x, y)+k-1 \geqq k-1$$

with equality if and only if $x=y$; hence $x \in \{b_2, \ldots, b_k\}$. For $y \notin b_2, \ldots, b_k$, the right hand side of (5.1) is at least

$$d(x, y)+d(b_1, y)+k-1 \geqq k-1$$

with equality only if $x=b_1=y$.

Now by Proposition 3.7,

$$f_\varrho(u, a_0) = d(u, a_0)+d(u, a_1)+OPT(u\, a_2 \ldots a_n a_0).$$

If $u \neq a_0$ then by Lemma 5.1 this is at least $n+1$. On the other hand if $u=a_0$ then Lemma 5.1 implies that $f_\varrho(a_0, a_0)=n$. This establishes the first part of the claim. The second part is proved analogously.

Next we show $WX(K_n) \leqq n$. Consider the following window n algorithm: given the current state s_i and the next n requests $r_{i+1}, \ldots, r_{i+n}$, if the first vertex repeated in the sequence $s_i\ r_{i+1} \ldots r_{i+n}$ is r_{i+1} then set $s_{i+1}=r_{i+1}$; otherwise set $s_{i+1}=s_i$. We claim this is optimal. Let s_0 be an initial state and $r_1 \ldots r_t$ be a request sequence, and let $s_1 \ldots s_t$ be the sequence given by the algorithm. We show, by induction on j that there exists an optimal sequence $a_1 a_2 \ldots a_t$ for $s_0\ r_1 \ldots r_t$ so that $a_i=s_i$ for $i \leqq j$. The basis case $j=0$ is vacuous. Now let $j \geqq 1$ and suppose $a_i=s_i$ for $i \leqq j-1$. If $a_j=s_j$ we're done, so assume $s_j \neq a_j$. The terms of $C(s_0 r_1 \ldots r_t;\ a_1 \ldots a_t)$ involving a_j are $d(a_{j-1}, a_j)+d(a_j, r_j)+d(a_j, a_{j+1})$. If $a_j \neq a_{j-1}$ or r_j then this is at least 2, so we can change a_j to s_j without increasing the cost (since s_j either equals a_{j-1} or r_j). So we can assume $a_j=s_{j-1}$ or $a_j=r_j$.

Case i. $a_j=a_{j-1}(=s_{j-1})$. Then $s_j=r_j$ so the first repeated vertex in $s_{j-1} r_j r_{j+1} \ldots r_{j+n-1}$ is r_j; assume this occurs at r_k. Let h be the smallest index exceeding j such that $a_h \neq s_{j-1}$. Then the cost of all terms involving $a_j a_{j+1} \ldots a_{\min(h-1,k)}$ can only decrease if we change them all from s_{j-1} to r_j.

Case ii. $a_j=r_j$. Then $s_j=a_{j-1}$, so the first vertex repeated is not r_j. Let r_k be the second occurrence of the first repeated vertex, and let h be the smallest index exceeding j such that $a_h \neq r_j$. Then the cost of all terms involving $a_j, a_{j+1}, \ldots, a_{\min(h-1,k)}$ can only decrease if we change them all from r_j to s_{j-1}.

Proof of Lemma 2.6. To prove

$$WX(G_1 \square G_2 \square \ldots \square G_K) = \max\big(WX(G_1)), \ldots, WX(G_K)\big),$$

it is enough to prove the result for the product of two graphs; the general case follows by induction. So, let $G=(V, E)$ and $H=(W, F)$. If $(v_0, w_0)(v_1, w_1)\ldots(v_t, w_t)$ is a DLP sequence in $G \square H$ then the cost of the state sequence $(x_1, y_1)(x_2, y_2)\ldots(x_t, y_t)$ in response is simply $C_G(v_0 v_1 \ldots v_t;\ x_1 \ldots x_t)+C_H(w_0 w_1 \ldots w_t;\ y_1 \ldots y_t)$ by the additivity of distances in $G \square H$. Hence, a response sequence is optimal if and only if both projections are optimal. Hence if G and H both have window k optimal algorithms then so does $G \square H$.

Proof of Lemma 2.7. Let $H=(W, F)$ be a retract of $G=(V, E)$ with retraction map f and let $W(G)=k$. Let $\varrho=a_0a_1...a_n$ be a DLP sequence for H. Then ϱ can be treated as a DLP sequence for G since H is an induced subgraph of G. Let $b_1b_2...b_n$ be an optimal state sequence in G for $a_0a_1...a_n$. Then we claim $f(b_1)f(b_2)...f(b_n)$ is an optimal state sequence in H for $a_0a_1...a_n$. This follows from the following sequence of inequalities:

$$OPT_H(a_0a_1...a_n) \geqq OPT_G(a_0a_1...a_n) = C_G(a_0a_1...a_n;\ b_1...b_n) \geqq$$
$$\geqq C_H(f(a_0)f(a_1)...f(a_n); f(b_1)...f(b_n)) =$$
$$= C_H(a_0a_1...a_n; f(b_1)...f(b_n)).$$

The first inequality holds since H is an induced subgraph of G. The second inequality holds since $d_G(v, w)\geqq d_H(f(v), f(w))$ by the property of retracts. The final equality follows from the fact that f is the identity on W.

Hence, if $b_1b_2...b_n$ is produced by a window k algorithm then so is $f(b_1)f(b_2)...f(b_n)$. ∎

6. Proof that (P1)→(P2)

Let G have finite windex. We will present four reductions that together immediately imply that G satisfies ($P2$). In each reduction, the finiteness of $WX(G)$ is shown to imply some local restriction on G. The proof of each reduction consists of producing a vertex sequence ϱ of length greater than $WX(G)$ and showing that if G doesn't conform to the restriction then the conclusion of Lemma 3.8 is contradicted.

Reduction 1. *Let a, b be nonadjacent vertices of G. Then $N(a)\cap N(b)$ is an independent set, i.e., G does not contain an induced subgraph on four vertices with exactly five edges.*

Proof. Suppose $c, d\in N(a)\cap N(b)$ and $\langle c, d\rangle\in E(G)$. Choose j such that $WX(G)<2j$ and define $\varrho=cd(ab)^j$.

Claim. *$f_\varrho(u, c)=OPT(\varrho c)$ only when $u=c$,*

and

$f_\varrho(u, d)=OPT(\varrho d)$ only when $u=d$,

which would contradict Lemma 3.8, and establish the reduction. We will prove the first claim; the second is completely analogous.

Lemma 6.1. *For $j\geqq 1$:*

(i) $OPT(x(ab)^jc)=2j$ *if* $x\in\{a, c\}$,

(ii) $OPT(x(ab)^jc)>2j$ *if* $x\notin\{a, c\}$.

First of all for any x, Lemma 3.5 (iv) gives

$$OPT(x(ab)^jc) \geqq jd(a, b) = 2j.$$

Furthermore

$$OPT(x(ab)^j c) \leqq C(x c^{2j};\ x(ab)^j c) = d(x,a)+d(x,c)+d(c,b)+2(j-1) =$$
$$= 2j \quad \text{if} \quad x\in\{a,c\},$$

proving (i).

We prove (ii) by induction on j. If $j=1$ then

$$OPT(xabc) \geqq \min_{v_1,v_2} d(x,v_1)+d(a,b)+d(v_2,c) \geqq 2+d(x,v_1)+d(v_2,c).$$

So assume $v_1=x$ and $v_2=c$ since otherwise we get a cost of at least 3. Then

$$OPT(xabc) = d(x,a)+d(x,c)+d(c,b) > 2$$

if $x\notin\{a,c\}$. For the induction step $j>1$ we have, by Proposition 3.7,

$$OPT(x(ab)^j c) = \min_y d(b,y)+OPT(xay)+OPT(y(ab)^{j-1}c).$$

By induction the third term is $>2j-2$ if $y\notin\{a,c\}$, and the first two terms sum to at least $d(a,b)=2$. So assume $y=a$ or c. If $y=c$ then $d(b,c)+ +OPT(xac)>2$ if $x\notin\{a,c\}$. If $y=a$ then $d(b,a)+OPT(xaa)=2+d(x,a)>2$ if $x\neq a$.

Now

$$f_\varrho(u,c) = d(u,c)+d(u,d)+OPT(u(ab)^j c).$$

If $u\notin\{a,c\}$ then Lemma 6.1 implies $f_\varrho(u,c)\geqq 2j+2$. For $u\in\{a,c\}$, Lemma 6.1 implies

$$f_\varrho(u,c) = d(u,c)+d(u,d)+2j$$

so $f_\varrho(u,c)$ has a unique minimum at $u=c$.

Reduction 2. *Let a, b be nonadjacent vertices of G. Then $|N(a)\cap N(b)|\leqq 2$.*

Proof. Suppose $c,d,e\in N(a)\cap N(b)$. By reduction 1, $\{c,d,e\}$ is an independent set. Choose j such that $WX(G)<3j$ and define $\varrho=(cde)^j$.

Claim. *$f_\varrho(u,a)=OPT(\varrho a)$ only when $u=a$, and $f_\varrho(u,b)=OPT(\varrho b)$ only when $u=b$*

which contradicts Lemma 3.8 and establishes the reduction. We will prove the first part of the claim; the second follows by symmetry.

First,

$$OPT(\varrho a) \leqq 3j$$

is shown by using the state sequence a^{3j-1}.

We show $f_\varrho(u,a)>3j$ for $u\neq a$ by induction on j. For $j=1$,

$$f_\varrho(u,a) = d(c,u)+d(d,u)+OPT(uea).$$

If $u=c$, d or e then this is at least 4. So assume $u\neq c, d, e$. Then the first two terms are each at least 1 and the last term is at least 2 unless $u=a$.

For the induction step $j>1$ we have by Proposition 3.7,

$$f_\varrho(u,a) = d(c,u)+d(d,u)+OPT(ue(cde)^{j-1}a) =$$
$$= \min_x d(c,u)+d(d,u)+d(u,x)+d(x,e)+OPT(x(cde)^{j-1}a).$$

Now if $x\neq a$ then the last term is at least $3j-2$ by the induction hypothesis and the first four terms are at least $OPT(cde)\geqq 3$, giving a cost of at least $3j+1$. So assume $x=a$, then the last term is $3j-1$ and the first four terms are

$$d(c,u)+d(d,u)+d(u,a)+d(a,e).$$

If $u=c$ or d then this sum is 4 and otherwise it is at least $3+d(u,a)$ and hence the total cost is at $3j+1$ if $u\neq a$.

Reduction 3. *Suppose* $a, b, c, e\in V(G)$ *and*

(i) $d(a,c)=d(b,c)=m$,

(ii) $d(e,c)=m+1$,

(iii) e *is adjacent to both* a *and* b *and* $d(a,b)=2$;

i.e., $\{a,b,c\}$ *is a type* 2 *spike. Then* a, b, c *have a median in* G.

Proof. Suppose a, b, c have no median in G. Let $\varrho=aec(bca)^j$ where j is chosen so that $3j>WX(G)$.

Claim. $f_\varrho(u,a)=OPT(\varrho a)$ *only when* $u=a$,
$f_\varrho(u,b)=OPT(\varrho b)$ *only when* $u=e$,
which would contradict Lemma 3.8, and establish the reduction.

Note first, by Proposition 3.6 and the assumption that b, c, a have no median

$$OPT(bca) = OPT(cab) = \min_x d(c,x)+d(a,x)+d(b,x) =$$
$$= m+2$$

(taking $x=a$).

Now using the response sequence a^{3j+3} on ϱa gives

$$OPT(\varrho a) \leqq (j+1)(m+2)-1.$$

On the other hand, using Proposition 3.7,

$$f_\varrho(u,a) = d(a,u)+d(u,e)+OPT(uc(bca)^j) \geqq$$
$$\geqq d(a,u)+d(u,e)+d(u,c)+j\,OPT(bca) \geqq$$
$$\geqq d(a,u)+d(c,e)+j(m+2) =$$
$$= d(a,u)+(j+1)(m+2)-1,$$

so $f_\varrho(u,a)$ is minimized precisely when $u=a$.

Next, using the response sequence eb^{3j} on ϱb gives

$$OPT(\varrho b) \leqq (j+1)(m+2).$$

On the other hand, using Proposition 3.7,

$$\begin{aligned} f_\varrho(u,b) &= d(a,u)+d(u,e)+OPT\big(uc(bca)^j b\big) = \\ &= d(a,u)+d(u,e)+OPT(ucb)+j(m+2) \geqq \\ &\geqq d(u,e)+OPT(acb)+j(m+2) \geqq \\ &\geqq d(u,e)+(j+1)(m+2) \end{aligned}$$

so $f_\varrho(u,b)$ is minimized uniquely when $u=e$.

Reduction 4. *Suppose a, b, c is a type 1 spike with $d(a,c)=d(b,c)=m$. Then a, b, c have an external median.*

Proof. Let $\varrho=acb(ca)^j$ where $2j>WX(G)$.

Claim. *$f_\varrho(u,c)=OPT(\varrho c)$ implies $u\in SP(c,a)\cap SP(c,b)$ and*

$f_\varrho(u,a)=OPT(\varrho a)$ implies u is a median of a, b, c.

Note that a and b are both medians but neither is in $SP(c,a)\cap SP(c,b)$ and $c\in SP(c,a)\cap SP(c,b)$ but is not a median, so by Lemma 3.8, the claim implies that a, b, c must have an external median.

Proof of claim. If u is in $SP(c,a)\cap SP(c,b)$ then evaluating the cost of the response sequence uc^{2j+2} for $u\varrho c$ gives

$$f_\varrho(u,c) \leqq (j+2)m \quad \text{for} \quad u\in SP(c,a)\cap SP(c,b).$$

On the other hand,

$$\begin{aligned} f_\varrho(u,c) &= d(u,a)+d(u,c)+OPT\big(ubc(ac)^j\big) \geqq \\ &\geqq d(u,a)+d(u,c)+OPT(u,b,c)+jm. \end{aligned}$$

Now $d(u,a)+d(u,c)\geqq m$ with equality only if $u\in SP(c,a)$ and $OPT(u,b,c)\geqq m$ with equality only if $u\in SP(c,b)$ so $f_\varrho(u,c)=(j+2)m$ if and only if $u\in SP(c,a)\cap\cap SP(c,b)$, establishing the first part of the claim.

For the second part, we evaluate the cost of the response a^{3j+3} to ϱa to get

$$f_\varrho(a,a) \leqq (j+1)m+1,$$

On the other hand,

$$\begin{aligned} f_\varrho(u,a) &= d(u,a)+d(u,c)+OPT\big(ub(ca)^j a\big) \geqq \\ &\geqq d(u,a)+d(u,c)+d(u,b)+jd(c,a) \geqq \\ &\geqq OPT(acb)+jm = \\ &= (j+1)m+1. \end{aligned}$$

Note that if u is not a median of a, b, c then the second inequality is strict, so $f_\varrho(u,a)>(j+1)m+1$ establishing the claim. ∎

7. The Proof of (P2) → (P3)

We first prove:

Theorem 7.1. *If G satisfies (P2) then G can be embedded isometrically in a product of cliques.*

A substantial theory of isometric embeddings has been developed by Djokovic [15], Firsov [18] and Graham and Winkler [24] (see [19] for a survey). We will need the following result from this theory.

Let G be any graph and define a binary relation R on the edges of G by:

$$\langle u, v\rangle R\langle u', v'\rangle \Leftrightarrow d(u, u')+d(v, v') \neq d(u, v')+d(v, u').$$

Recall that if $F \subseteq E(G)$ then G/F denotes the graph obtained by contracting the edges of F. More precisely each connected component of F is a vertex of G/F with two components adjacent if some edge of $E-F$ joins them.

Theorem 7.2 ([24]). *Let $E_1, \ldots, E_s$ be the partition of E into its connected R-components. Then*

$$G \xrightarrow{I} G/(E-E_1)\Box G/(E-E_2)\Box \ldots \Box G/(E-E_s).$$

Hence to show that G embeds isometrically into a product of cliques it is enough to show that $G/(E-E_i)$ is a clique for each i. Our aim is to construct the classes E_i.

Throughout this section G denotes a graph that satisfies $(P2)$.

Let Q denote the set of all vertex subsets that induce maximal complete subgraphs in G, for $\mathcal{Q} \in Q$, $E_{\mathcal{Q}}$ denotes the set of edges induced on $\mathcal{Q}$.

Lemma 7.3. *Each edge of G belongs to a unique maximal complete subgraph, hence $\{E_{\mathcal{Q}} | \mathcal{Q} \in Q\}$ is a partition of the edge set.*

Proof. Let $\langle a, b\rangle \in E$. If $c, d \in N(a) \cap N(b)$ then by $(P2\text{a})$, $\langle c, d\rangle \in E$. Thus, $\{a, b\} \cup \cup(N(a) \cap N(b))$ induces a complete subgraph in G and it clearly contains any other set A that contains $\{a, b\}$ and induces a complete subgraph in G.

For each $\langle a, b\rangle \in E$ we denote by $\mathcal{Q}(a, b)$ the unique member of Q containing a and b. By the above proof:

$$\mathcal{Q}(a, b) = \{a, b\} \cup (N(a) \cap N(b)). \tag{7.1}$$

Lemma 7.4. *Let $\mathcal{Q} \in Q$. Then for each $a \in V$ there is a unique $b \in \mathcal{Q}$ such that $d(a, b) = = d(a, \mathcal{Q})$.*

Proof. Suppose there were two vertices $b, c \in \mathcal{Q}$ such that $d(a, b) = d(a, c) = d(a, \mathcal{Q})$. Then a, b, c form a type 1 spike and so by $(P2\text{b})$, have a median e. Then $e \in N(b) \cap \cap N(c)$ and hence, by (7.1), $e \in \mathcal{Q}(b, c) = \mathcal{Q}$. Then $d(a, e) = d(a, b) - 1 = d(a, \mathcal{Q}) - 1$, a contradiction.

We denote by $\alpha_{\mathcal{Q}}(a)$ the unique vertex b in $\mathcal{Q}$ such that $d(a, b) = d(a, \mathcal{Q})$.

Lemma 7.5. *Let $\mathcal{Q}, \mathcal{Q}' \in Q$. Then exactly one of the following holds:*

(i) $|\mathcal{Q}|=|\mathcal{Q}'|$ *and the elements of* $\mathcal{Q}$ *and* $\mathcal{Q}'$ *can be labelled* $\{a_1, \ldots, a_r\}$ *and* $\{b_1, \ldots, b_r\}$ *such that*

$$d(a_i, b_j) = \begin{cases} d(\mathcal{Q}, \mathcal{Q}') & \text{if } i = j \\ d(\mathcal{Q}, \mathcal{Q}')+1 & \text{if } i \neq j \end{cases};$$

(ii) *There are unique vertices* $a \in \mathcal{Q}$, $b \in \mathcal{Q}'$ *so that* $d(a, b)=d(\mathcal{Q}, \mathcal{Q}')$, *and for* $v \in \mathcal{Q}'-b$, $w \in \mathcal{Q}-a$,

$$d(a, v) = d(w, b) = d(\mathcal{Q}, \mathcal{Q}')+1$$

and

$$d(w, v) = d(\mathcal{Q}, \mathcal{Q}')+2.$$

Proof. Let $m=d(\mathcal{Q}, \mathcal{Q}')$ and let $A \subseteq \mathcal{Q}$ be the set of vertices $v \in \mathcal{Q}$ such that $d(v, \mathcal{Q}')=d(\mathcal{Q}, \mathcal{Q}')$. Define $A' \subseteq \mathcal{Q}'$ analogously. Then $\alpha_{\mathcal{Q}}$ maps A' to A and $\alpha_{\mathcal{Q}'}$ maps A to A' and these maps must be inverses. So if $A=\mathcal{Q}$ and $A'=\mathcal{Q}'$ then (i) holds. Otherwise, suppose $b \in \mathcal{Q}'-A'$. Then $d(b, \mathcal{Q})=m+1=d(b, v)$ for any $v \in A$. By Lemma 7.4, A must consist of one element a. Similarly A' consists of one element b. For $v \in \mathcal{Q}'-b$, $w \in \mathcal{Q}-a$, $d(a, v)=d(w, b)=d(\mathcal{Q}, \mathcal{Q}')+1$ and $d(v, w)= =d(\mathcal{Q}, \mathcal{Q}')+2$ by Lemma 7.4.

We now define a relation on Q: $\mathcal{Q}|\mathcal{Q}'$ if $\mathcal{Q}$ and $\mathcal{Q}'$ satisfy conclusion (i) of Lemma 7.5. Clearly $\mathcal{Q}|\mathcal{Q}'$ if and only if $\mathcal{Q}'|\mathcal{Q}$. We will prove

Lemma 7.6. $\mathcal{Q}|\mathcal{Q}'$ *and* $\mathcal{Q}'|\mathcal{Q}''$ *implies* $\mathcal{Q}|\mathcal{Q}''$, *and hence* $|$ *is an equivalence relation on* Q.

To prove this we need two preliminary observations.

Lemma 7.7 *If* $\mathcal{Q}, \mathcal{Q}' \in Q$ *and there exist* $a_1, a_2 \in \mathcal{Q}$ *such that* $d(a_1, \mathcal{Q}')=d(a_2, \mathcal{Q}')= =d(\mathcal{Q}, \mathcal{Q}')$ *then* $\mathcal{Q}|\mathcal{Q}'$.

Proof. The hypothesis violates condition (ii) of Lemma 7.5; hence condition (i) holds, i.e., $\mathcal{Q}|\mathcal{Q}'$.

Lemma 7.8. *Let* $\mathcal{Q}_0, \mathcal{Q}_m \in Q$ *with* $d(\mathcal{Q}_0, \mathcal{Q}_m)=m$ *and* $\mathcal{Q}_0|\mathcal{Q}_m$. *Then there exist* $\mathcal{Q}_1, \ldots, \mathcal{Q}_{m-1} \in Q$ *such that* $\mathcal{Q}_i|\mathcal{Q}_{i+1}$ *with* $d(\mathcal{Q}_i, \mathcal{Q}_{i+1})=1$ *for* $0 \leqq i \leqq m-1$.

Proof. We proceed by induction on m; for $m=1$ the result is trivial. Let $\mathcal{Q}_0= =\{a_1, \ldots, a_r\}$ and $\mathcal{Q}_m=\{b_1, \ldots, b_r\}$ with $d(a_i, b_j)=m$ if and only if $i=j$. Let c_1 be a first element along a shortest path from b_1 to a_1. Then $d(c_1, b_1)=1=d(c_1, \mathcal{Q}_m)$ and $d(c_1, a_1)=m-1=d(c_1, \mathcal{Q}_0)$ and, by Lemma 7.4, if $j \neq 1$, then $d(b_j, c_1)=2$ and $d(a_j, c_1)=m$. Then $\{c_1, a_2, b_2\}$ form a (type 2) spike and hence by (P2b) have a median $c_2 \in N(b_2) \cap N(c_1)$ with $d(c_2, a_2)=m-1$. Let $\mathcal{Q}_{m-1}=\mathcal{Q}(c_1, c_2)$. Then $b_j \notin \mathcal{Q}_{m-1}$ for $j>1$ since $d(c_1, b_j)>1$ and $b_1 \notin \mathcal{Q}_{m-1}$ since $\{b_1, c_2\} \subseteq N(b_2) \cap N(c_1)$ and (P2a) imply $\langle b_1, c_2 \rangle \notin E$. Hence $\mathcal{Q}_{m-1} \cap \mathcal{Q}_m=\emptyset$ so $d(\mathcal{Q}_{m-1}, \mathcal{Q}_m)=d(c_1, b_1)= =d(c_2, b_2)=1$. Thus by Lemma 7.5, $\mathcal{Q}_{m-1}|\mathcal{Q}_m$ and so $\mathcal{Q}_{m-1}=\{c_1, c_2, \ldots, c_r\}$ with $d(b_i, c_j)=1$ if and only if $i=j$. Now $d(\mathcal{Q}_0, \mathcal{Q}_{m-1}) \geqq m-1$ since $d(\mathcal{Q}_0, \mathcal{Q}_m)=m$ and so $d(\mathcal{Q}_0, \mathcal{Q}_{m-1})=m-1=d(c_1, a_1)=d(c_2, a_2)$. Hence $\mathcal{Q}_0|\mathcal{Q}_{m-1}$ by Lemma 7.7 and we can apply the induction hypothesis to complete the proof.

Proof of Lemma 7.6. Assume $\mathcal{Q}|\mathcal{Q}''$ does not hold; we will derive a contradiction. Let $m=d(\mathcal{Q}, \mathcal{Q}')$. By Lemma 7.8, there is a sequence $\mathcal{Q}'=\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_m=\mathcal{Q}$ such that $\mathcal{Q}_i|\mathcal{Q}_{i+1}$ and $d(\mathcal{Q}_i, \mathcal{Q}_{i+1})=1$. Let j be the largest index such that $\mathcal{Q}_j|\mathcal{Q}''$. By assumption $j<m$. Then $\mathcal{Q}_{j+1}|\mathcal{Q}''$ does not hold. Let $p=d(\mathcal{Q}_j, \mathcal{Q}'')$. Let $\mathcal{Q}_{j+1}=\{a_1, ..., a_r\}$, $\mathcal{Q}_j=\{b_1, ..., b_r\}$ and $\mathcal{Q}''=\{c_1, ..., c_r\}$ labelled so that $\langle a_i, b_i\rangle\in E$ and $d(b_i, c_i)=p$ for $1\leq i\leq k$. We distinguish the three cases $d(\mathcal{Q}_{j+1}, \mathcal{Q}'')=p-1$, p and $p+1$.

Case i. $d(\mathcal{Q}_{j+1}, \mathcal{Q}'')=p+1$. Then $d(a_i, c_i)=d(\mathcal{Q}_{j+1}, \mathcal{Q}'')$ for all i and so $\mathcal{Q}_{j+1}|\mathcal{Q}''$ by Lemma 7.7.

Case ii. $d(\mathcal{Q}_{j+1}, \mathcal{Q}'')=p-1$. Then $d(a_i, c_h)=p-1$ for some $a_i\in\mathcal{Q}_{j+1}$, $c_h\in\mathcal{Q}''$. Then $d(b_i, c_h)\leq p$, so $i=h$. Assume $i=1$; by Lemma 7.7 it is enough to show $d(a_2, c_2)=p-1$. Now $d(a_1, c_2)=d(b_2, c_2)=p$ and $d(b_1, c_2)=p+1$ with $b_1\in N(a_1)\cap N(b_2)$; hence $\{a_1, b_2, c_2\}$ is a (type 2) spike and (P2b) implies there exists $v\in N(a_1)\cap N(b_2)$ with $d(v, c_2)=p-1$. Then $b_1, a_2, v\in N(a_1)\cap N(b_2)$ so (P2a) implies $v=a_2$ and $d(a_2, c_2)=p-1$, as required.

Case iii. $d(\mathcal{Q}_{j+1}, \mathcal{Q}'')=p$; without loss of generality $d(a_1, \mathcal{Q}'')=p$.

Subcase a. $d(a_1, c_1)=p$. If $d(a_2, \mathcal{Q}'')=p$ then Lemma 7.7 implies $\mathcal{Q}_{j+1}|\mathcal{Q}''$ so we can assume $d(a_2, c_1)=d(a_2, c_2)=p+1$. Then $\{a_2, c_1, c_2\}$ is a (type 1) spike and (P2a) implies there is a vertex $e\in N(c_1)\cap N(c_2)$ with $d(a_2, e)=p$. But then $e\in\mathcal{Q}(c_1, c_2)=\mathcal{Q}''$ and $d(a_2, \mathcal{Q}'')=p$ and again Lemma 7.7 implies $\mathcal{Q}_{j+1}|\mathcal{Q}''$.

Subcase b. $d(a_1, c_i)=p$ for some $i\neq 1$. Then $\{a_1, b_i, c_i\}$ is a spike since $b_1\in N(a_1)\cap N(b_i)$ and $d(b_1, c_i)=p+1$, so by (P2b) there is a vertex $e\in N(a_1)\cap N(b_i)$ with $d(e, c_i)=p$. Then $b_1, a_i, e\in N(a_1)\cap N(b_j)$, so by ($P$2a) $e=a_i$. Then $p-1=d(a_i, \mathcal{Q}'')\geq d(\mathcal{Q}_{j+1}, \mathcal{Q}'')$ contradicting the case assumption. ▌

Therefore, $|$ is an equivalence relation; let $[\mathcal{Q}]$ denote the equivalence class of $\mathcal{Q}$, and let $E_{[\mathcal{Q}]}$ denote $\bigcup_{\mathcal{Q}'\in[\mathcal{Q}]} E_{\mathcal{Q}}$. Then the $E_{[\mathcal{Q}]}$ partition the edges.

Lemma 7.9. *The R-components of E are the sets* $E_{[\mathcal{Q}]}$.

This is an immediate consequence of the next three lemmas.

Lemma 7.10. *The edges of any clique are in the same R-component, hence* $E_{\mathcal{Q}}$ *is contained in an R-component for each* $\mathcal{Q}\in Q$.

Proof. For any $u, v, w\in\mathcal{Q}$, $\langle u, v\rangle R\langle u, w\rangle$.

Lemma 7.11. *If* $\mathcal{Q}|\mathcal{Q}'$ *then* $E_{\mathcal{Q}}$ *and* $E_{\mathcal{Q}'}$ *are in the same R-component.*

Proof. If $a_1, a_2\in\mathcal{Q}'$ with $b_1=\alpha_{\mathcal{Q}}(a_1)$ and $b_2=\alpha_{\mathcal{Q}}(a_2)$ then by a property of $|$, $d(a_1, b_1)+d(a_2, b_2)<d(a_1, b_2)+d(a_2, b_1)$ and so $\langle a_1, a_2\rangle R\langle b_1, b_2\rangle$.

Lemma 7.12. *If* $e\in E_{\mathcal{Q}}$, $e'\in E_{\mathcal{Q}'}$ *and* eRe', *then* $\mathcal{Q}|\mathcal{Q}'$.

Proof. If $\mathcal{Q}|\mathcal{Q}'$ does not hold, then the conditions of Lemma 7.5 (ii) hold, which imply that $d(v, v')+d(w, w')=d(v, w')+d(w, v')$ for all $v, w\in\mathcal{Q}$ and $v', w'\in\mathcal{Q}'$.

In light of Lemma 7.9, Theorem 7.1 will follow if we show $G/(E-E_{[\mathcal{Q}]})$ is a clique for each class $[\mathcal{Q}]$.

Lemma 7.13. *If* $\mathcal{Q}|\mathcal{Q}'$ *and* $a\in\mathcal{Q}$, *then there is a path from a to* $\alpha_{\mathcal{Q}'}(a)$ *consisting only of edges in* $E-E_{[\mathcal{Q}]}$.

Proof. Let $d(\mathcal{Q}, \mathcal{Q}')=m$ and let $\mathcal{Q}=\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_m=\mathcal{Q}'$ be the sequence given by Lemma 7.8. Then the sequence $a_0=a, a_1, \ldots, a_m=\alpha_{\mathcal{Q}'}(a)$ defined by $a_i=\alpha_{\mathcal{Q}_i}(a_{i-1})$ is a path from a to $\alpha_{\mathcal{Q}'}(a)$. Moreover $\mathcal{Q}(a_{i-1}, a_i)\notin[\mathcal{Q}]$ since $\mathcal{Q}_i\in[\mathcal{Q}]$ and the members of $[\mathcal{Q}]$ are disjoint. Hence the path lies in $E-E_{[\mathcal{Q}]}$.

Finally, let $V_1, V_2, \ldots, V_q$ be the partition of V into the connected components of $G'=(V, E-E_{[\mathcal{Q}]})$. Then in $G/(E-E_{[\mathcal{Q}]})$, each V_j contracts to a vertex. Hence it is enough to show that any two of them have an edge of $E_{[\mathcal{Q}]}$ joining them. Since G is connected, each V_j contains a vertex of some member of $[\mathcal{Q}]$, but then by Lemma 7.13, V_j intersects every member of $[\mathcal{Q}]$ and thus any two of them have an edge of $E_{[\mathcal{Q}]}$ between them. ∎

Having established Theorem 7.1, we can now assume that G is an isometric subgraph of B_W where $W=W_1\times W_2\times\ldots\times W_r$. So, let $G=B_V$ with $V\subseteq W$. We want to show that V is imprint closed. Let $\bar{a}, \bar{b}, \bar{c}\in V$; we want to show that $imp(\bar{a}, \bar{b}:\bar{c})\in V$. Let $I\subseteq\{1, 2, \ldots, r\}$ be the set of indices for which $a_i\neq b_i$. We proceed by induction on $|I|$. We first write $I=I_0\cup I_a\cup I_b$ where I_a is the set of indices for which $a_i=c_i$, I_b is the set of indices for which $b_i=c_i$ and $I_0=I-(I_a\cup I_b)$. Note that if $I=I_a$ then $\bar{a}=imp(\bar{a}, \bar{b}:\bar{c})$ and if $I=I_b$ then $\bar{b}=imp(\bar{a}, \bar{b}:\bar{c})$.

If $|I|=1$, we need only consider the case $I=I_0$. Then $d(\bar{a}, \bar{c})=d(\bar{b}, \bar{c})$ and so $(\bar{a}, \bar{b}, \bar{c})$ is a spike. The only median besides $\bar{a}$ and $\bar{b}$ is $imp(\bar{a}, \bar{b}:\bar{c})$ which is in V by (P2a).

If $|I|=2$, let $I=\{i, j\}$. Since B_V is an isometric subgraph, $\bar{a}$ and $\bar{b}$ have a common neighbor $\bar{d}$ in V with $d_i=a_i$ and $d_j=b_j$. Let $\bar{a}'=imp(\bar{a}, \bar{d}:\bar{c})$ and $\bar{b}'=imp(\bar{b}, \bar{d}:\bar{c})$. By the previous case $\bar{a}', \bar{b}'\in V$. If $i\in I_a$ then $\bar{a}'=imp(\bar{a}, \bar{b}:\bar{c})$ and if $j\in I_b$ then $\bar{b}'=imp(\bar{a}, \bar{b}:\bar{c})$. Otherwise $\bar{a}', \bar{b}'$ and $\bar{c}$ form a type 2 spike with $\bar{d}\in N(\bar{a}')\cap N(\bar{b}')$ and $d(\bar{c}, \bar{d})=d(\bar{c}, \bar{a}')+1$. Hence $imp(\bar{a}, \bar{b}:\bar{c})=imp(\bar{a}', \bar{b}':\bar{c})\in V$ by (P2a).

Finally, if $|I|>2$ then let $\bar{d}$ be the first vertex on a shortest path from $\bar{a}$ to $\bar{b}$, and $\bar{e}$ be the first vertex on same shortest path from $\bar{b}$ to $\bar{a}$. Let $\bar{a}'=imp(\bar{a}, \bar{e}:\bar{c})$ and $\bar{b}'=imp(\bar{b}, \bar{d}:\bar{c})$. Then by induction $\bar{a}', \bar{b}'\in V$. Furthermore, $imp(\bar{a}, \bar{b}:\bar{c})=imp(\bar{a}', \bar{b}':\bar{c})$. But $d(\bar{a}', \bar{b}')=2$ so by induction, this is in V. ∎

8. Proof of (P3) → (P4)

Let V be an imprint-closed subset of $W=W_1\times W_2\times\ldots\times W_r$ such that B_V is connected (we may assume V is irreducible and W is the product completion of V). We want to show that B_V is a retract of B_W.

Define $C(W)$, the *coordinate set* of W, to be the set of all pairs (a, i) where $a\in W_i$. For $(a, i)\in C(W)$, we define $S(a, i)$ to be the set of $\bar{w}\in W$ such that $w_i=a$; $S(a, i)$ is called a *slice* of W. If (a, i), (b, j) are coordinates with $i\neq j$, then $L((a, i), (b, j))=S(a, i)\cup S(b, j)$; it is called an *L-set*. The set $\{i, j\}$ is called the *support* of the L-set. We will first prove:

Theorem 8.1. *If V is imprint-closed and B_V is connected then V is an intersection of L-sets of W.*

Proof. It is enough to show that for each $\bar{w}\in W-V$ there is an L-set that contains V but not $\bar{w}$. Let $I\subseteq\{1, \ldots, r\}$ be a minimal subset such that $proj_I(\bar{w})\notin proj_I(V)$. Then $|I|>1$ by the irreducibility of W. If $|I|\geqq 3$, let $i_1, i_2, i_3\in I$ and let $\bar{v}^1, \bar{v}^2, \bar{v}^3\in V$

be such that $proj_{I-i_j}(\bar{v}^j)=proj_{I-i_j}(\bar{w})$. Then $\bar{v}=imp(\bar{v}^1, \bar{v}^2:\bar{v}^3)$ satisfies $\bar{v}\in V$ and $proj_I(\bar{v})=proj_I(\bar{w})$, a contradiction; so $|I|=2$. Assume without loss of generality that $I=\{1, 2\}$. Now by the irreducibility of W, there are vertices $\bar{a}, \bar{b}\in V$ such that $a_1=w_1$ and $b_2=w_2$. We claim $L((b_1, 1), (a_2, 2))$ contains V; this will prove the theorem since $\bar{w}\notin L((b_1, 1), (a_2, 2))$.

We consider the set $proj_I(V)$ as the edge set of a bipartite graph Γ on $W_1\times W_2$. The connectivity of B_V implies that Γ is connected. Now if $\bar{c}, \bar{d}\in V$ we have $c_1=d_1$ and $c_2\neq d_2$ then $proj_I(imp(\bar{c}, \bar{d}:\bar{x}))=(c_1, x_2)$ for any $\bar{x}\in V$. Hence by the irreducibility of V, each vertex in W_1 is adjacent in Γ to either exactly one vertex or to all vertices in W_2 (and similarly with W_1 and W_2 reversed). These conditions, together with the facts that (a_1, a_2), (b_1, b_2) are in Γ but (a_1, b_2) is not in Γ, imply that every edge in Γ is incident on either b_1 in W_1 or a_2 in W_2 and hence $V\subseteq L((b_1, 1), (a_2, 2))$.

To prove $(P3)\to(P4)$ it is now enough to show:

Theorem 8.2. *Let V be a connected subset of $W=W_1\times W_2\times\ldots\times W_r$ (where V is irreducible and W is the product completion of V). If V is an intersection of L-sets then B_V is a retract of B_W.*

Proof. Let (a, i), (b, j) be coordinates with $i\neq j$. Define the map $f=f^{(a,i)(b,j)}: W\to W$ as follows: f fixes a vector $\bar{w}$ if $w_i=a$ or $w_j=b$ and otherwise $f(\bar{w})$ is obtained from $\bar{w}$ by changing w_i to a and w_j to b. It is easy to see:

Lemma 8.3. *$f^{(a,i)(b,j)}$ is a retraction of W onto $L((a, i), (b, j))$.*

We call $f^{(a,i)(v,j)}$ the *elementary retraction* to $L((a, i), (b, j))$.

Now define a graph P on the vertex set $\{1, 2, \ldots, r\}$ with edges $\langle i, j\rangle$ if V is contained in an L-set with support $\{i, j\}$.

Lemma 8.4. *For each $\langle i, j\rangle\in E(P)$ there is a unique L-set containing V with support $\{i, j\}$.*

Proof. Suppose $V\subseteq L((a, i), (b, j))$. Since V is irreducible there are vertices $\bar{c}, \bar{d}\in V$ such that $c_i\neq a$ and $d_j\neq b$, and hence $c_i=b$ and $d_j=a$. Since V is connected there is a path from $\bar{c}$ to $\bar{d}$ in B_V. Let $\bar{e}$ be the first vertex on the path with $e_i=a$; then the previous vertex on the path has jth coordinate equal to b (since it is in $L((a, i), (b, j))$ so also $e_j=b$. Now the only L-set with support $\{i, j\}$ containing $\bar{c}$, $\bar{d}$, $\bar{e}$ is $L((a, i), (b, j))$.

By Lemma 8.4, for each $\langle i, j\rangle\in E(P)$ there are unique elements $a_i(j)\in W_i$ and $a_j(i)\in W_j$ such that $V\subseteq L((i, a_i(j), (j, a_j(i)))$. Let f_{ij} denote the corresponding elementary retract, and let $L_{ij}=L((i, a_i(j)), (j, a_j(i)))$. Set $F=\{f_{ij}|\langle i, j\rangle\in E(P)\}$.

Lemma 8.5. *For each $\bar{v}\in W$ there exists a sequence of elementary retracts in F whose composition maps $\bar{v}$ to V.*

Before proving this lemma, let us note that it implies that any superset V' of V can be retracted to V and hence implies Theorem 8.2. To see this, proceed by induction on $|V'|$; if $|V'|=|V|$ the result is trivial. So assume $|V'|>|V|$ and let $\bar{v}\in V'-V$. The sequence of retracts given by Lemma 8.5 fixes V (by Lemma 8.3) and maps $\bar{v}$ to V and hence maps V' to another superset V'' of V having strictly smaller cardinality than V'. Now apply the induction hypothesis to V''.

Proof of Lemma 8.5. We first note:

Lemma 8.6. *If* $\langle i,j\rangle, \langle j,k\rangle \in E(P)$ *and* $a_j(i) \neq a_j(k)$ *then* $\langle i,k\rangle \in E(P)$ *with* $a_k(i) = a_k(j)$ *and* $a_i(k) = a_i(j)$.

Proof. The hypotheses imply that every $\bar{v} \in V$ satisfies either $v_i = a_i(j)$ or $v_j = a_j(i)$ and either $v_j = a_j(k)$ or $v_k = a_k(j)$. Therefore $a_j(i) \neq a_j(k)$ implies either $v_i = a_i(j)$ or $v_k = a_k(j)$.

For $\langle i,j\rangle \in E(P)$, say $\bar{v}$ *satisfies* $\langle i,j\rangle$ if $\bar{v} \in L_{ij}$. We proceed by induction on the number of unsatisfied edges; if there are none then $\bar{v} \in V$. So assume there is at least one unsatisfied edge.

A satisfied edge $\langle i,j\rangle$ is critical if not both $v_i = a_i(j)$ and $v_j = a_j(i)$. We orient each critical edge by $i \to j$ if $v_i = a_i(j)$ and $v_j \neq a_j(i)$. (Note $i \to j$ means that if we change v_i and leave v_j fixed then $\langle i,j\rangle$ becomes unsatisfied.) By Lemma 8.6 we have:

Lemma 8.7. (i) $i \to j$ *and* $j \to k$ *implies* $i \to k$;
(ii) *if* $\langle i,j\rangle$ *is unsatisfied and* $i \to k$ *then* $\langle j,k\rangle$ *is an unsatisfied edge of* $E(P)$.

From (i) and the fact that each edge has a unique orientation we have:

Lemma 8.8. $\to$ *is a transitive acyclic relation, i.e., a partial order, on* $\{1, \ldots, r\}$.

Let us call an element $h \in \{1, \ldots, r\}$ a *sink* if no edge is oriented away from it. Now let $\langle i,j\rangle$ be any unsatisfied edge. By Lemma 8.8 and a basic property of partial order relations there is a sink h with either $i = h$ or $i \to h$. Then Lemma 8.7 (ii) implies $\langle h,j\rangle$ is an unsatisfied edge in $E(P)$. Similarly there is a sink k with either $j = k$ or $j \to k$ and, again by Lemma 8.7 (ii), $\langle h,k\rangle$ is an unsatisfied edge in $E(P)$. Then $f_{h,k}(\bar{v})$ satisfies $\langle h,k\rangle$. Moreover since h and k are sinks, $f_{hk}(\bar{v})$ satisfies any edge that $\bar{v}$ does. Hence $f_{hk}(\bar{v})$ has fewer unsatisfied edges and we can apply the induction hypothesis to complete the proofs of Lemma 8.5 and Theorem 8.2. ∎

9. Computing $WX(G)$

To complete the proof of Theorem 3.1 we show

Lemma 9.1. *If* $WX(G)$ *is finite then* $WX(G) = \Theta(G)$.

Proof. If $W \subseteq V(G)$ induces a clique then G_W is always retract of G and so by Proposition 2.3 and Lemma 2.7, $WX(G) \geq WX(G_W)$ so $WX(G) \geq \Theta(G)$. Now if $WX(G)$ is finite then $(P3)$ says that $G = B_V$ where V is a connected imprint closed subset of $W_1 \times \ldots \times W_r$ (where $W_1 \times \ldots \times W_r$ is irreducible). Then we claim for each i, $\Theta(G) \geq |W_i|$.

If $a, b \in W_i$ then by the irreducibility of W, there are vectors $\bar{v}, \bar{w} \in V$ with $v_i = a$ and $w_i = b$; by the connectivity of V we can choose such $\bar{v}$ and $\bar{w}$ that agree in all other positions. Then $\{imp(\bar{v}, \bar{w} : \bar{y}) | \bar{y} \in V\}$ forms a clique of size $|W_i|$ in V.

Finally the proof of $(P3) \to (P4)$ shows B_V is a retract of B_W so by Lemmas 2.6 and 2.7 and Proposition 2.3, $WX(B_V) \leq \max_{1 \leq i \leq r} \{|W_i|\} \leq \Theta(G)$.

If we want to determine $WX(G)$ for a given graph we can do either of the following.

A) Check whether G satisfies $P2$. If not, $WX(G)=\infty$. If so, then since every edge is in a unique maximal clique, $\Theta(G)$ is easily computed.

B) Construct the R-equivalence classes $E_1, \dots, E_r$ of $E(G)$. If $G/(E-E_i)$ is not a clique for some i then $WX(G)=\infty$. Otherwise $WX(G)= =\max |V(G/(E-E_i))|$.

10. Remarks and Open Problems

A) Graphs of Windex 2

It is not difficult to show that, for triangle free graphs, $(P2)$ is equivalent to USTP. Furthermore, as was observed by Bandelt, every weak retract of a hypercube is obtainable by a strong retraction map. Hence in this case we can refine Theorem 3.1:

Theorem 10.1. *The following conditions on a connected graph G are equivalent:*

(Q1) $WX(G)=2$;

(Q2) *G has USTP;*

(Q3) *G is a median-closed subgraph of a hypercube;*

(Q4) *G is a weak retract of a hypercube;*

(Q5) *G is a strong retract of a hypercube.*

The equivalence of (Q2) through (Q5) were already known and characterize a well-studied class of graphs called median graphs (see [33], [34], [4]).

B) Optimal Waste Ratios

Since "most" graphs do not have finite windex, it is natural to consider how well a window-k algorithm can perform on a given graph. If A is any algorithm for the DLP on a graph G and ϱ is a problem instance, let $C_A(\varrho)$ be the cost of the response sequence produced by algorithm A for ϱ. For instance if A is the (window 1) algorithm that defines the ith response to be equal to the ith request then it is easy to show that for any input ϱ,

$$C_A(\varrho) \leqq 2 \cdot OPT(\varrho).$$

We propose:

Conjecture 10.2. *Let G be any graph and k be an integer* $\geqq 1$. *Then there exists an algorithm A for the DLP on G that operates within window k such that on any problem instance* ϱ,

$$C_A(\varrho) \leqq \left(1+\frac{1}{k}\right) OPT(\varrho).$$

C) The Search Value of a Graph

The *average value* of an instance $a_0 a_1 \dots a_n$ of the DLP on G is defined to be $AV(a_0 a_1 \dots a_n) = OPT(a_0 a_1 \dots a_n)/n$. In [12], we defined the search value $\lambda(G)$ to be the worst case average cost of an arbitrarily long request sequence. More precisely, if $\lambda_n(G) = \max_{a_0 a_1 \ldots a_n} AV(a_0 a_1 \ldots a_n)/n$ then

$$\lambda(G) = \limsup_{n \to \infty} \lambda_n(G).$$

Bounds on $\lambda(G)$ are obtained in [12], it is unknown whether $\lambda(G)$ is polynomially computable.

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References

[1] P. Assouad, Un espace hypermétrique non plongeable dans un espace L, *C. R. Acad. Sci. Paris,* **286** (ser. A) (1977), 361—363.
[2] P. Assouad and C. Delorme, Graphs plongeables dans L, *C. R. Acad. Sci. Paris,* **291** (1980), 369—372.
[3] S. P. Avann, Metric ternary distributive semi-lattices, *Proc. Amer. Math. Soc.,* **12** (1961), 407—414.
[4] H. J. Bandelt, Retracts of hypercubes, *preprint.*
[5] H. J. Bandelt and J. Hedlíková, Median algebras, *Discrete Math.* **45** (1983), 1—30.
[6] J. R. Bitner, Heuristics that dynamically alter data structure to reduce theirs access time, Ph. D. Thesis, Univ. of Illinios, (1976).
[7] J. R. Bitner, Heuristics that dynamically organize data structures, *SIAM J. Comp.,* **8** (1979), 82—110.
[8] A. Borodin, N. Linial and M. Saks, An optimal online algorithm for metrical task systems, *Proc. 19th ACM Symposium on Theory of Computing,* 1987 (to appear).
[9] H. J. Bandelt and H. M. Mulder, Infinite mediam graphs, (0,2)-graphs and hypercubes. *J. Graph Theory* (to appear).
[10] J. L. Bentley, C. C. McGeoch, Worst-case analyses of self-organizing sequential search heuristics, *Proc.* 20th *Allerton Conference on Communication, Control and Computing.*
[11] F. R. K. Chung, J. Cohen and R. L. Graham, Pursuit-evasion in graphs, *Journal of Graph Theory* **12** (1980), 159—167.
[12] F. R. K. Chung, R. L. Graham and M. E. Saks, Dynamic Search in Graphs, in Discrete Algorithms and Complexity, Academic Press (1987), 351—388.
[13] F. R. K. Chung, D. J. Hajela and P. Seymour, Self-organizing sequential search and Hilbert's inequality, Proc. 17th ACM Symposium on Theory of Computing (1985), 217—223.
[14] A. K. Dewdney, The embedding dimension of a graph, *Ars Combinatoria,* **9** (1980), 77—90.
[15] D. Z. Djoković, Distance preserving subgraphs of hypercubes, *J. Comb. Th.* (B), **14** (1973), 263—267.
[16] D. Duffus and I. Rival, Graphs orientable as distributive lattices, *Proc. Amer. Math. Soc.* **88** (1983), 197—200.
[17] E. Evans, Median lattices and convex subalgebras, Colloq. Math. Soc. János Bolyai, 29 *Universal algebra* (1982), 225—240.
[18] V. Firsov, Isometric embedding of a graph in a Boolean cube, *Kibernetica (Kiev),* **1** (1965), 95—96.
[19] R. L. Graham, On isometric embeddings of graphs, in Proc. of Silver Jubilee Conference on Combinatorics, Waterloo University, 1984.
[20] R. L. Graham, On primitive graphs and optimal vertex assignments, *NY Acad. Sci.* **175** (1970), 170—186.
[21] G. Gonnet, J. I. Munro and H. Suwanda, Toward self-organizing sequential search heuristics, Proc 20th IEEE Symp. Foundations Computer-Science, (1979), 169—174.
[22] R. L. Graham and H. O. Pollak, On the addressing problem for loop switching, *Bell Sys. Tech. Jour.,* **50** (1971), 2495—2519.
[23] R. L. Graham and H. O. Pollak, On embedding graphs in squashed cubes, *Graph Theory and Applications,* in Lecture Notes in Math. No. 303, Springer-Verlag, New York, 1972, 99—110.

[24] R. L. GRAHAM and P. M. WINKLER, On isometric embeddings of graphs, *Transactions Amer. Math. Soc.*, **288** (1985), 527—539.
[25] P. HELL, Rétractions de graphes, Ph. D. thesis, Université de Montréal, 1972.
[26] P. HELL, Absolute planar retracts and the four color conjecture, *J. Combinatorial Theory*, **17** (1974), 5—10.
[27] P. HELL, Absolute retracts in graphs, Springer-Verlag, New York, Lecture Notes Math., **406** (1974), 291—301.
[28] P. HELL, Graph retractions, Colloq. Intern. Teorie Combinatories II, Roma, 1976, 263—268.
[29] J. R. ISBELL, Mediam algebra, *Trans. Amer. Math. Soc.*, **260** (1980), 319—362.
[30] D. E. KNUTH, The Art of Computer Programming Vol. 3, Sorting and Searching, Addison-Wesley, Reading, MA (1973), 398—399.
[31] Y. C. KAN and S. M. ROSS, Optimal list under partial memory constraints, *J. Appl. Prob.*, **17** (1980), 1004—1015.
[32] J. MCCABE, On serial file with relocatable records, *Oper. Res.*, **12** (1965), 609—618.
[33] H. M. MULDER, The structure of median graphs, *J. Graph Theory*, **4** (1980), 107—110.
[34] H. M. MULDER, *n*-Cubes and median graphs, *J. Graph Theory*, **4** (1980), 107—110.
[35] H. M. MULDER and A. SCHRIJVER, Median graphs and Helly hypergraphs. *Discrete Math.*, **25** (1979), 41—50.
[36] L. NEBESKÝ, Median graphs, *Comment. Math., Univ. Carolina*, **12** (1971), 317—325.
[37] J. NIEMINEN, Join-semilattices and simple graphic algebras, *Math. Nachr.*, **77** (1977), 87—91.
[38] R. NOWAKOWSKI and I. RIVAL, Fixed-edge theorem for graphs with loops, *J. Graph Theory*, 3 (1979), 339—350.
[39] R. NOWAKOWSKI and I. RIVAL, On a class of isometric subgraphs of a graph, *Combinatorica*, 2 (1982), 79—90.
[40] R. NOWAKOWSKI and I. RIVAL, The smallest graph variety containing all paths, *Discrete Math.*, **43** (1983), 223—234.
[41] R. NOWAKOWSKI and P. WINKLER, Vertex-to-vertex pursuit in a graph, *Discrete Math.* **43** (1983), 235—239.
[42] C. PAPADIMITRIOU and K. STEIGLITZ, *Combinatorial Optimization*, Prentice Hall, 1982.
[43] R. RIVEST, On self-organizing sequential search heuristics, *CACM*, **19** (1976), 63—67.
[44] D. D. SLEATOR and R. E. TARJAN, Amortized efficiency of list update and paging rules, *Comm. ACM* **28** (1985), 205—208.
[45] R. E. TARJAN, Amortized computational complexity, *SIAM, J. Alg. Disc. Meth.*, **6** (1985), 306—318.

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